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On knotted real projective planes

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ABSTRACT

Let $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ be a hyperbolic transformation. Let B be a new band attaching to L such that $L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}}$ is also a hyperbolic transformation. In this paper, we will study the relationship between the realizing surfaces $F(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ and $F(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$. If B is a noncoherent band to both L and $L_{\mathfrak{B}}$ such that $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ is defined, then $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}) \sharp \mathbf{R}P^2$ and $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}}) \sharp \mathbf{R}P^2$ are ambient isotopic, where $\mathbf{R}P^2$ is one of the standard real projective planes. We will study the triviality of $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ because as an application, $\mathbf{R}P^2$ can untangle some knotted sphere $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ with suitable conditions, when it is attached to $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ by the connected sum.

Keywords: Hyperbolic transformation; realizing surface; closed realizing surface; standard real projective planes; ribbon knot; ribbon 2-knot.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

A *surface-link* is a closed surface F embedded in $\mathbf{R}^4$ locally flatly. If F is connected, it is called a *surface-knot*. A surface-knot F is called a 2-*knot* if it is a 2-sphere. A surface-link F is called a 2-*link* if each component of F is a 2-sphere.

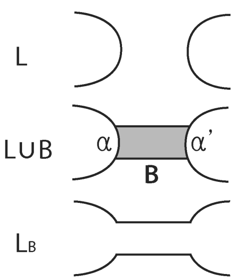
There are many descriptions for surface-links, such as broken surface diagrams, movies (or motion pictures), charts, decker sets and so on, see details in [3, 9]. In [13], Kawauchi, Shibuya and Suzuki introduced a hyperbolic transformation which can give the motion picture description to describe an orientable surface link in $\mathbf{R}^4$.

Let L be a link in $\mathbf{R}^3$. A *band attaching to* L is a 2-disk B in $\mathbf{R}^3$ if $L \cap B = L \cap \partial B = \{\alpha, \alpha'\}$ where α and α' are disjoint arcs in ∂B. Then $L_B = cl((L \cup \partial B) - (L \cap B))$ is a link, which is called *the link obtained from* L *by a hyperbolic transformation* $L \xrightarrow{B} L_B$ *along* B. A *band set attaching to* L is a set $\mathfrak{B} = \{B_1, B_2, \ldots, B_n\}$ of mutually disjoint bands $B_1, B_2, \ldots, B_n$ attaching to L. Then $L_{\mathfrak{B}} = cl((L \cup \partial(\bigcup_{k=1}^n B_k)) - (L \cap (\bigcup_{k=1}^n B_k)))$ is a link, which is called *the link obtained from* L *by a hyperbolic transformation* $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ *along* $\mathfrak{B}$.

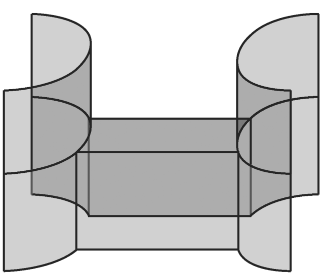
For a given hyperbolic transformation $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$, define a proper surface F in $\mathbf{R}^3[a,b]$ by

$$F \cap \mathbf{R}^3[t] = \begin{cases} L, & t \in \left[a, \dfrac{a+b}{2}\right), \\ L \cup \left(\displaystyle\bigcup_{B \in \mathfrak{B}} B\right), & t = \dfrac{a+b}{2}, \\ L_{\mathfrak{B}}, & t \in \left(\dfrac{a+b}{2}, b\right]. \end{cases}$$

The resulting surface F is called the *realizing surface* of $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ in $\mathbf{R}^3[a,b]$ and is denoted by $F_{[a,b]}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$. Note that $F_{[a,b]}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is a knotted surface with boundary $L \cup L_{\mathfrak{B}}$. If L and $L_{\mathfrak{B}}$ are trivial links with μ-components and μ'-components, respectively, then we can get a knotted surface $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ without boundary from $F_{[a,b]}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ by attaching $(\mu + \mu')$ mutually disjoint disks to L and $L_{\mathfrak{B}}$ whose construction is known to give a unique surface-link up to ambient isotopic, see [13]. We call $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ the *closed realizing surface* of $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ in $\mathbf{R}^4$ (or a *closure* of $F_{[a,b]}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$).



A hyperbolic transformation $L \xrightarrow{B} L_B$



The realizing surface of $L \xrightarrow{B} L_B$

It is well-known that the Euler characteristic χ of F is $\mu + \mu' - n$ where μ, μ' and n denote the number of components of L and $L_{\mathfrak{B}}$ and the number of bands of $\mathfrak{B}$, respectively. The genus g of $F = F_{[a,b]}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is $1 - \frac{1}{2}(\mu + \mu' - n)$ if F is orientable or g is $2 - (\mu + \mu' - n)$ if F is non-orientable.

Proposition 1.1 ([13]). *Every orientable surface-link is ambient isotopic to the closed realizing surface of a hyperbolic transformation.*

Let B be a band attaching to a link L with μ-components and μ' the number of components of L_B. A band B is called a *fission* band if $\mu' = \mu + 1$. A band B is called a *fusion* band if $\mu' = \mu - 1$. Let $\mathfrak{B}$ be a band set attaching to L consisting of n bands. A hyperbolic transformation $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ is said to be *n-fission* if every band in $\mathfrak{B}$ is a fission band. A hyperbolic transformation $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ is said to be *n-fusion* if every band in $\mathfrak{B}$ is a fusion band. In particular, if L is a knot and if $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ is n-fission, then $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ is said to be *complete fission*. If $L_{\mathfrak{B}}$ is a knot and if $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ is n-fusion, then $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ is said to be *complete fusion*.

It is clear that if a hyperbolic transformation $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ is n-fusion, then the inverse $L_{\mathfrak{B}} \xrightarrow{\mathfrak{B}} L$ of $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ is n-fission, that is, the fusion and the fission are dual concepts. Originally, these concepts are introduced by Hosokawa [4].

Definition 1.2 ([13]). An orientable surface-knot F in $\mathbf{R}^4$ is said to be *in a normal form* if F is the closed realizing surface of a sequence $O_- \xrightarrow{\mathfrak{B}_{--}} K_- \xrightarrow{\mathfrak{B}_-} L \xrightarrow{\mathfrak{B}_+} K_+ \xrightarrow{\mathfrak{B}_{++}} O_+$ with the following properties.

- O_- and O_+ are trivial links.
- K_- and K_+ are knots and L is a link.
- $O_- \xrightarrow{\mathfrak{B}_{--}} K_-$ and $L \xrightarrow{\mathfrak{B}_+} K_+$ are complete fusion.
- $K_- \xrightarrow{\mathfrak{B}_-} L$ and $K_+ \xrightarrow{\mathfrak{B}_{++}} O_+$ are complete fission.

The link L is called *the middle cross-sectional link of* F and K_- (respectivley, K_+) *the lower* (respectively, *upper*) *cross-sectional knot*. In particular, if L is a knot in $\mathbf{R}^3$, then we have $K_- = L = K_+$.

Proposition 1.3 ([13]). *Every orientable surface-knot F is ambient isotopic to a surface-knot in a normal form. In particular, every 2-knot is ambient isotopic to the surface-knot of the form $\hat{F}(L_s \xrightarrow{\mathfrak{B}_{--}} K \xrightarrow{\mathfrak{B}_{++}} L_t)$ where K is a knot, $L_s \xrightarrow{\mathfrak{B}_{--}} K$ is complete fusion and $K \xrightarrow{\mathfrak{B}_{++}} L_t$ is complete fission. We call $\hat{F}(L_s \xrightarrow{\mathfrak{B}_{--}} K \xrightarrow{\mathfrak{B}_{++}} L_t)$ the normal form of a 2-knot.*

In 1989, Kamada [8] modified the definition of a normal form for non-orientable surface-knots in $\mathbf{R}^4$ and proved that every non-orientable surface-knot in $\mathbf{R}^4$ is ambient isotopic to a surface-knot in a normal form. In order to define the *normal form* for non-orientable surface-knots, he introduced the definition of a *coherent* band and a *noncoherent* band.

Let B be a band attaching to a link L with μ-componenets and μ' the number of components of L_B. A band B is said to be *coherent* if $\mu' = \mu + 1$ or $\mu' = \mu - 1$. A band B is said to be *noncoherent* if $\mu' = \mu$. Note that any band attaching to distinct components of L is coherent. If a band B is attached to one component

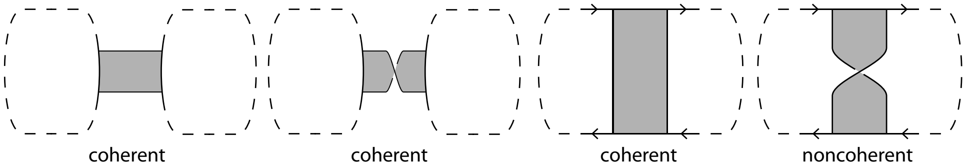


Fig. 1. Coherent bands and a noncoherent band.

of L, then B is coherent if and only if the orientation of B agrees with a given orientation of the component of L, in other words, B is noncoherent if and only if the orientation of B does not agree with a given orientation of the component of L, see Fig. 1.

Let K be a knot and $\mathfrak{B}$ a set of mutually disjoint noncoherent bands $B_1, B_2, \ldots, B_n$ attaching to K. A set $\mathfrak{B}$ is said to be *in regular position* to K if there exist mutually disjoint n simple arcs $I_1, I_2, \ldots, I_n$ on K such that for each $j = 1, 2, \ldots, n$, the attaching arcs of B_j are contained in I_j, see Fig. 2.

Definition 1.4 ([8]). A non-orientable surface-knot F in $\mathbf{R}^4$ is said to be *in a normal form* if F is the closed realizing surface of a sequence $O_- \xrightarrow{\mathfrak{B}_-} K_- \xrightarrow{\mathfrak{B}_\circ} K_+ \xrightarrow{\mathfrak{B}_+} O_+$ with the following properties.

- O_- and O_+ are trivial links.
- $O_- \xrightarrow{\mathfrak{B}_-} K_-$ is complete fusion.
- $K_+ \xrightarrow{\mathfrak{B}_+} O_+$ is complete fission.
- $\mathfrak{B}_\circ$ is a set of noncoherent bands and is in regular position to K_-.

We call $K_- \cup \mathfrak{B}_\circ = K_+ \cup \mathfrak{B}_\circ$ *the middle cross-section of* F and K_- (respectively, K_+) *the lower* (respectively, *upper*) *cross-sectional knot.*

Proposition 1.5 ([8]). *Every non-orientable surface-knot F in $\mathbf{R}^4$ is ambient isotopic to a surface-knot in a normal form. If $F = \hat{F}(O_- \xrightarrow{\mathfrak{B}_-} K_- \xrightarrow{\mathfrak{B}_\circ} K_+ \xrightarrow{\mathfrak{B}_+} O_+)$, then the genus of F is equal to the number of bands of $\mathfrak{B}_\circ$.*

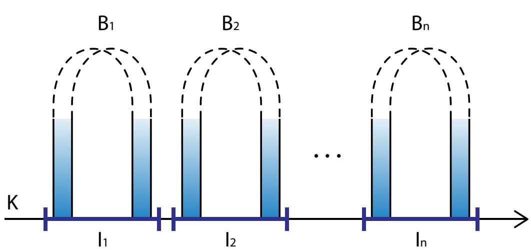


Fig. 2. Noncoherent bands are in regular position to K.

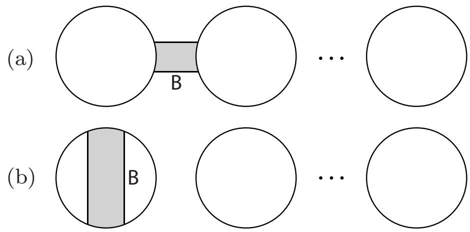


Fig. 3.

A set $\mathfrak{B}$ of coherent bands attaching to the trivial link L is said to be *standard* if $L \cup (\cup\mathfrak{B})$ is planar. For example, if a fusion band B is standard, then $L \cup B$ is homeomorphic to the union of the link and the band in Fig. 3(a) or if a fission band B is standard, then $L \cup B$ is homeomorphic to the union of the link and the band in Fig. 3(b).

In [10, 11], Kawauchi showed that there exist infinitely many non-standard fusion band sets. In the paper, he gave an example of a nontrivial fusion band set $\mathfrak{B} = \{B_1, B_2\}$ attaching to the trivial 3-component link as depicted in Fig. 4, which is a modification of Howie and Short's example given in [7]. Notice that two bands B_1 and B_2 in Fig. 4 are linked each other.

Let N be a 3-ball by attaching two 1-handles and let D_v and D_h be two disks satisfying $D_v \perp D_h$ in Fig. 5(a). Let $a_1, a_2, \ldots, a_m$ be simple closed curves in ∂N, parallel to ∂D_v in Fig. 5(b). Let $b_1, b_2, \ldots, b_n$ be simple closed curves in ∂N,

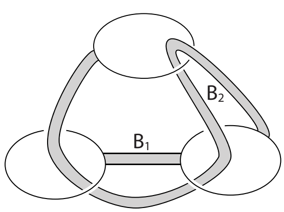


Fig. 4.

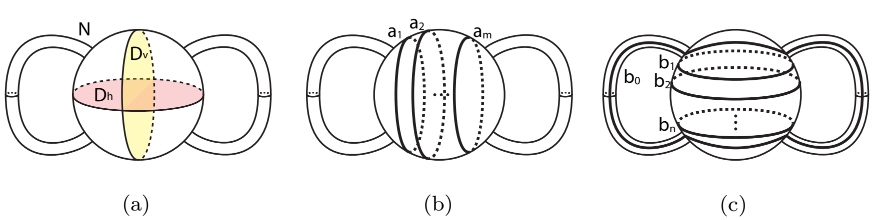


Fig. 5.

parallel to ∂D_h and b_0 a simple closed curve which passes once over each 1-handle as depicted in Fig. 5(c).

Proposition 1.6 ([16]). *Suppose N is embedded in an oriented 3-manifold M in such a way that some $A_m = \{a_1, a_2, \ldots, a_m\}$ (m odd) and some $B_n = \{b_0, b_1, \ldots, b_n\}$ bound planar surfaces P and Q in $\overline{M-N}$. Then A_1 and some B_0 bound disks in $\overline{M-N}$ which intersect in a single arc.*

Lemma 1.7. *Let B be a fusion band attaching to a μ-component link L in $\mathbf{R}^3$ ($\mu \geq 2$). If L and L_B are trivial, then B is standard.*

Proof. Scharlemann proved that the result holds for $\mu = 2$. The following proof is a modification of Scharlemann's proof for general case.

Let $L = O_1 \cup O_2 \cup \cdots \cup O_\mu$ be the trivial μ-component link in $\mathbf{R}^3$ and B a fusion band to L. Then there exist two components of L attached by B, say O_1 and O_2. Then $L_B = (O_1 \cup O_2)_B \cup O_3 \cup \cdots \cup O_\mu$ where $(O_1 \cup O_2)_B$ is the unknot obtained from O_1 and O_2 by a hyperbolic transformation along B. Consider a 3-manifold $M \equiv \mathbf{R}^3 \backslash (O_3 \cup \cdots \cup O_\mu)$. Notice that $(O_1 \cup O_2)_B$ is unknotted in M and $O_1 \cup O_2$ is a split link in M. Let N be a regular neighborhood of $O_1 \cup O_2 \cup B$ in M. Let P' be a 2-sphere in M which separates O_1 and O_2 such that $P' \cap N$ is the union of m disks (m odd) whose boundary is A_m and let Q' be a disk bounded by $(O_1 \cup O_2)_B$ such that $Q' \cap N$ is the union of n disks whose boundary is B_n. Put $P = P' \cap \overline{M-N}$ and $Q = Q' \cap \overline{M-N}$. Then P and Q are two planar surfaces with boundary A_m and B_n respectively. By Proposition 1.6, A_1 bounds a disk in $\overline{M-N}$ and hence one can obtain a 2-sphere which separates O_1 from O_2 in M such that the intersection of the 2-sphere and the band B is a single arc. Hence B is standard. $\square$

A noncoherent band B attaching to the trivial link L is said to be *standard* if $L \cup B$ is homeomorphic to the union of circles and a *standard* half-twisted band, as in Fig. 6. Notice that the band B in Fig. 6(a) is right-handed while the band B in Fig. 6(b) is left-handed.

Bleiler and Scharlemann modified Scharlemann's results for noncoherent bands.

Let N be a 3-ball by attaching two 1-handles and let D_v and D_h be two disks satisfying $D_v \perp D_h$ as depicted in Fig. 7(a). Let $a_1, a_2, \ldots, a_m$ be simple closed

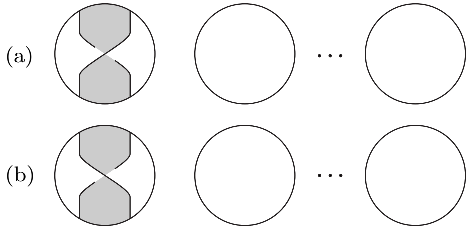
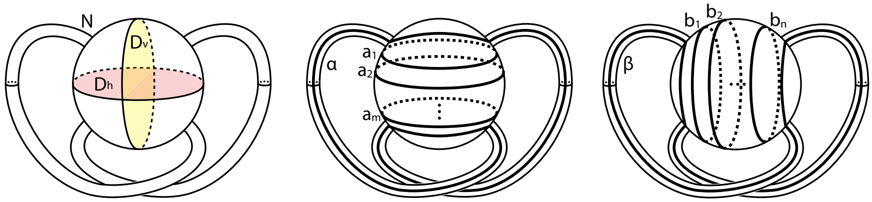


Fig. 6.



(a) (b) (c)

Fig. 7.

curves in ∂N, parallel to ∂D_h and $b_1, b_2, \ldots, b_n$ simple closed curves in ∂N, parallel to ∂D_v. Let α and β be simple closed curves which passes once over each 1-handle illustrated in Fig. 7(b) and Fig. 7(c), respectively.

Proposition 1.8 ([2]). *Suppose that N is embedded in an oriented 3-manifold M so that some $A_m = \{\alpha, a_1, a_2, \ldots, a_m\}$ and some $B_n = \{\beta, b_1, b_2, \ldots, b_n\}$ bound embedded planar surfaces P and Q in $\overline{M-N}$. Then some A_0 and B_0 bound embedded disks E_P and E_Q in $\overline{M-N}$ and either*:

(1) *E_P and E_Q are disjoint and $\mathbf{R}P^3$ is a summand of M or*
(2) *there is an embedded disk D in $\overline{M-N}$, disjoint from E_P and E_Q, such that ∂D is the union along the boundary of two arcs, one crossing a 1-handle once, and the other lying in the boundary of the 3-ball.*

Lemma 1.9. *Let B be a noncoherent band attaching to a μ-component link L in $\mathbf{R}^3$. If L and L_B are trivial, then B is standard.*

Proof. The following proof is obtained by modifying Bleiler and Scharlemann's proof for $\mu = 1$.

Let $L = O_1 \cup O_2 \cup \cdots \cup O_\mu$ be the trivial μ-component link in $\mathbf{R}^3$ and B a noncoherent band to L. Then there exists one component of L attached by B, say O_1. Then $L_B = (O_1)_B \cup O_2 \cup \cdots \cup O_\mu$. Consider a 3-manifold $M \equiv \mathbf{R}^3 \backslash (O_2 \cup \cdots \cup O_\mu)$. Let N be a regular neighborhood of $O_1 \cup B$ in M. Notice that $(O_1)_B$ is unknotted in M. Let E_P be a disk bounded by O_1 such that $E_P \cap N$ is the union of m disks whose boundary is A_m and let E_Q be a disk bounded by $(O_1)_B$ such that $E_Q \cap N$ is the union of n disks whose boundary is B_n. Put $P = E_P \cap \overline{M-N}$ and $Q = E_Q \cap \overline{M-N}$. Then P and Q are two planar surfaces with boundary A_m and B_n respectively. Since M is not a summand of $\mathbf{R}P^3$, by Proposition 1.8, there exists an embedded disk D in $\overline{M-N}$, disjoint from E_P and E_Q, such that ∂D is the union along the boundary of two arcs, one crossing a 1-handle, and the other lying in the boundary of the 3-ball. $E_P \cup B$ is an I-bundle on ∂D with p half-twists. Then $(O_1)_B$ is a $(2, p)$-torus knot. Since $(O_1)_B$ is unknotted, $p = 1$ or $p = -1$. Hence B is standard. □

2. Main Results

Let $L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}$ be a hyperbolic transformation. Let B be a band attaching to L such that $\mathfrak{B} \cup \{B\}$ is mutually disjoint. Let L_B and $L_{\mathfrak{B}\cup\{B\}}$ denote the links obtained from L by a hyperbolic transformation along B and $\mathfrak{B} \cup \{B\}$, respectively. Note that $L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}}$ is a hyperbolic transformation.

Lemma 2.1. *If B is a noncoherent band to both L and $L_{\mathfrak{B}}$, then $F(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ and $F(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ are homeomorphic surfaces.*

Proof. Since B is a noncoherent band to both L and $L_{\mathfrak{B}}$, the links L_B and $L_{\mathfrak{B}\cup\{B\}}$ have the same number of components as L and $L_{\mathfrak{B}}$, respectively. By calculating the Euler characteristic of $F(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ and $F(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$, one can see that $F(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ and $F(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ have the same genus. □

Remark 2.2. In general, $F(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ and $F(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ are not ambient isotopic. For, there is a counter example which was introduced by Viro [17]. Consider the trivial 2-component link L and the band set $\mathfrak{B} = \{B_1, B_2\}$ attaching to L in Fig. 8(a). Note that if B_1 is a fusion band, then B_2 is a fission band, and that $L_{\mathfrak{B}}$ is trivial. Notice that the closed realizing surface $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is a knotted 2-knot, whose motion picture is given by the left side figure in Fig. 8(b). In fact, the fundamental group of the complement of $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is given as $\langle x, y \,|\, xy^2x^{-2}y^{-1}x^2y^{-2} = 1\rangle$ and the Alexander polynomial of $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is $t^2 - 2$, see [17]. Hence $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is nontrivial.

On the other hand, the band B in Fig. 8(a) is a noncoherent band attaching to L. Since $\mathfrak{B} \cup \{B\}$ is mutually disjoint and since L_B and $L_{\mathfrak{B}\cup\{B\}}$ are trivial, one can get the new closed realizing surface $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$, whose motion picture is given by the right side figure in Fig. 8(b). In Sec. 3, we will show that the 2-knot $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ is trivial.

It is well-known that there are two standard embeddings of the real projective plane in $\mathbf{R}^4$ whose motion pictures are illustrated in Fig. 9, see [1, 3, 4, 6, 8]. The Euler number of the real projective plane in Fig. 9(a) is 2, while the Euler number of the real projective plane in Fig. 9(b) is -2 so that they are not ambient isotopic [18]. We call both of them the *standard real projective planes* $\mathbf{R}P^2$, and denote them by $\mathbf{R}P^2_+$ if the Euler number of $\mathbf{R}P^2$ is 2, and by $\mathbf{R}P^2_-$ if the Euler number of $\mathbf{R}P^2$ is -2.

From now on, we will show that $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}) \sharp \mathbf{R}P^2$ and $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}}) \sharp \mathbf{R}P^2$ can be ambient isotopic even though $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ and $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ are not ambient isotopic.

Theorem 2.3. *Let $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ denote a surface-knot. Let B be a band attaching to L such that $\mathfrak{B} \cup \{B\}$ is mutually disjoint and that L_B and $L_{\mathfrak{B}\cup\{B\}}$ are all trivial*

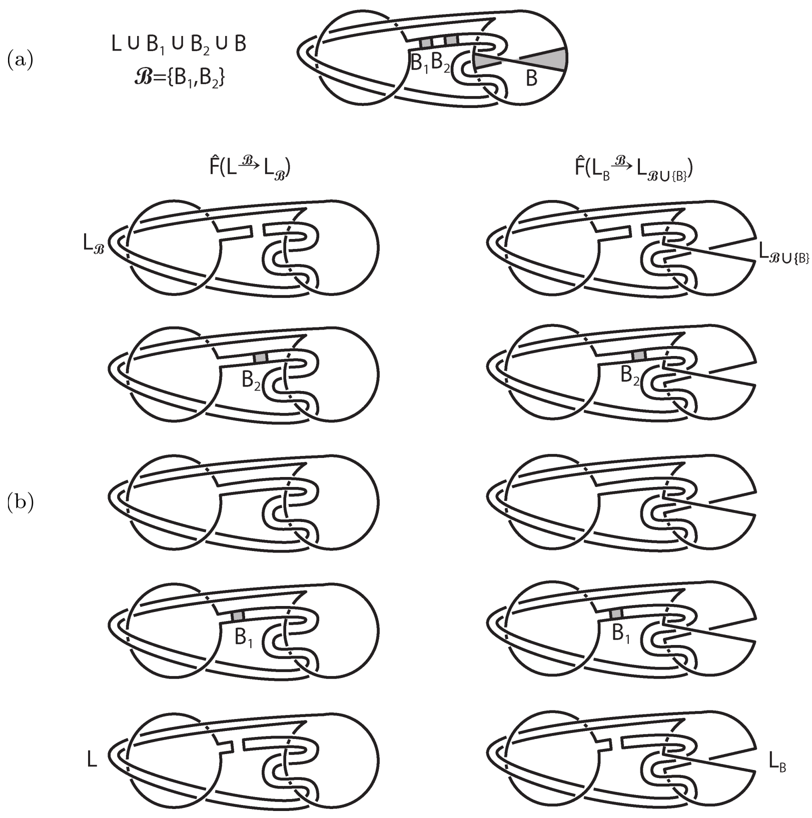


Fig. 8. Viro's example.

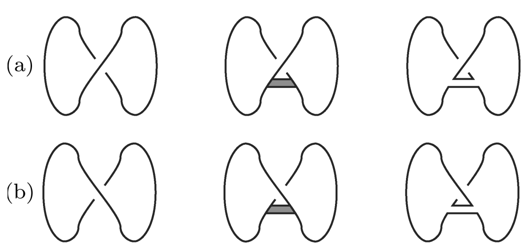


Fig. 9. Motion pictures of the standard real projective planes.

so that $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ *is defined. If* B *is a noncoherent band to both* L *and* $L_{\mathfrak{B}}$, *then* $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}) \sharp \mathbf{R}P^2$ *and* $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}}) \sharp \mathbf{R}P^2$ *are ambient isotopic, where*

$$\mathbf{R}P^2 = \begin{cases} \mathbf{R}P^2_+ & \text{if } B \text{ is right-handed}, \\ \mathbf{R}P^2_- & \text{if } B \text{ is left-handed}. \end{cases}$$

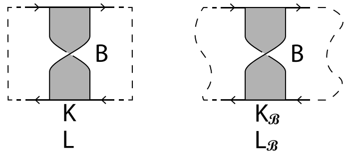


Fig. 10.

Proof. Since B is a noncoherent band to both L and $L_{\mathfrak{B}}$ and since L_B and $L_{\mathfrak{B}\cup\{B\}}$ are trivial links, B is standard to both L and $L_{\mathfrak{B}}$ by Lemma 1.9.

Suppose that B is right-handed. Since B is a standard noncoherent band to L, there exists a component K of L such that B is attaching to K, see Fig. 10(a). Similarly, there exists a component $K_{\mathfrak{B}}$ of $L_{\mathfrak{B}}$ such that B is attaching to $K_{\mathfrak{B}}$, see Fig. 10(b).

If the motion picture of $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is of the form in the top of Fig. 11, then the motion picture of $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ is of the form in the bottom of Fig. 11, where two motion pictures are exactly the same except the local part that B is attached. Without loss of generality, we may assume that any band in $\mathfrak{B}$ does not appear in Fig. 11 because $\mathfrak{B} \cup \{B\}$ is mutually disjoint.

Let $\mathbf{R}P^2_+$ be the standard real projective plane in Fig. 9 with the Euler number 2. Notice that $\mathbf{R}P^2_+$ is the closed realizing surface of the hyperbolic transformation $O \xrightarrow{B_0} O'$ in Fig. 12.

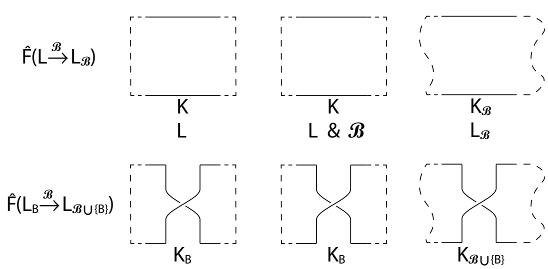


Fig. 11.

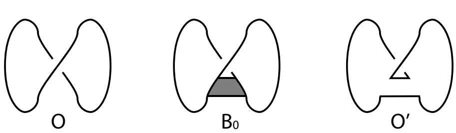


Fig. 12.

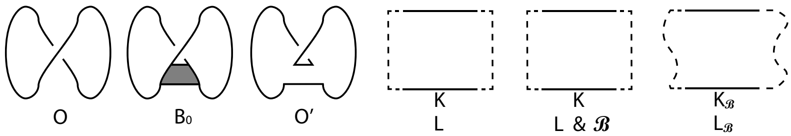


Fig. 13.

Since $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is connected, $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ is also connected so that one can attach $\mathbf{R}P^2_+$ to $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ in any preferred place. Since O' is the unknot and L contains the trivial component K, we can make the connected sum of $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ and $\mathbf{R}P^2_+$ by identifying O' and K as seen in Fig. 13.

Similarly, since O is the unknot and $L_{\mathfrak{B}\cup\{B\}}$ contains the trivial component $K_{\mathfrak{B}\cup\{B\}}$, the connected sum of $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ and $\mathbf{R}P^2_+$ is given as in Fig. 14.

We claim that $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}) \sharp \mathbf{R}P^2_+$ and $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}}) \sharp \mathbf{R}P^2_+$ are ambient isotopic by giving a movie of the motion pictures. In Fig. 15, the motion picture (A)

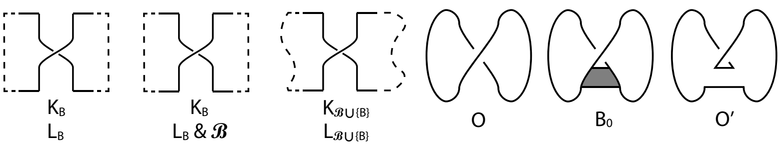


Fig. 14.

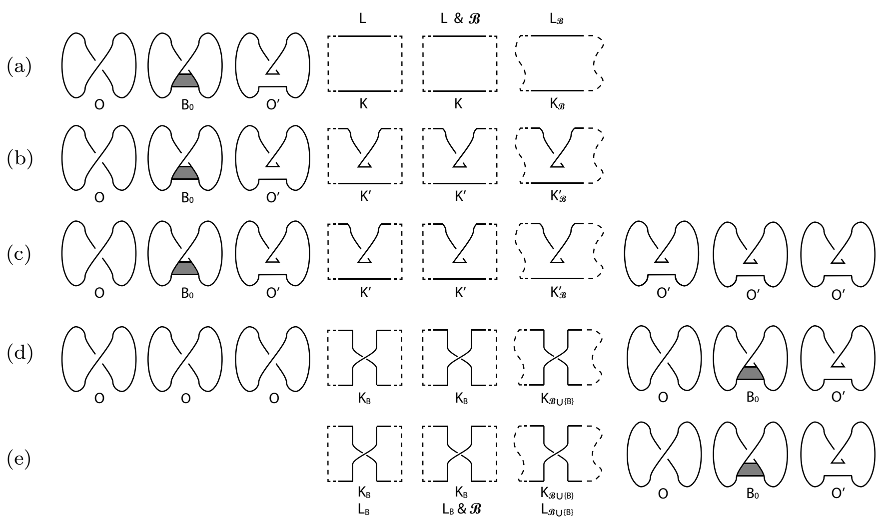


Fig. 15.

is $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}) \sharp \mathbf{R}P^2_+$. The motion picture (B) is obtained from (A) by applying Reidemeister move I. Since $K'_{\mathfrak{B}}$ is trivial, by a suitable ambient isotopy, $K'_{\mathfrak{B}}$ can be deformed to O'. Hence we get the motion picture (C). By pushing up the band B_0 through the movie, we get the motion picture (D) which is ambient isotopic to the motion picture (E). In fact, the motion picture (E) is $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}}) \sharp \mathbf{R}P^2_+$. Hence $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}) \sharp \mathbf{R}P^2_+$ and $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}}) \sharp \mathbf{R}P^2_+$ are ambient isotopic.

The proof for the case that B is left-handed is similar. $\square$

Remark 2.4. (1) The condition that $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is a surface-knot is necessary to define the connected sum of $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ and $\mathbf{R}P^2$ uniquely.

(2) For the closed realizing surface $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ and a new band B attaching to L, the links L_B and $L_{\mathfrak{B}\cup\{B\}}$ are nontrivial even though L and $L_{\mathfrak{B}}$ are all trivial. For example, consider the trivial 2-component link L and the band set $\mathfrak{B} = \{B_1, B_2\}$ attaching to L in Fig. 16(a). Note that if B_1 is a fusion band, then B_2 is a fission band and that $L_{\mathfrak{B}}$ is trivial. Consider the noncoherent band B attaching to L in Fig. 16(a). Notice that $\mathfrak{B}\cup\{B\}$ is mutually disjoint. However,

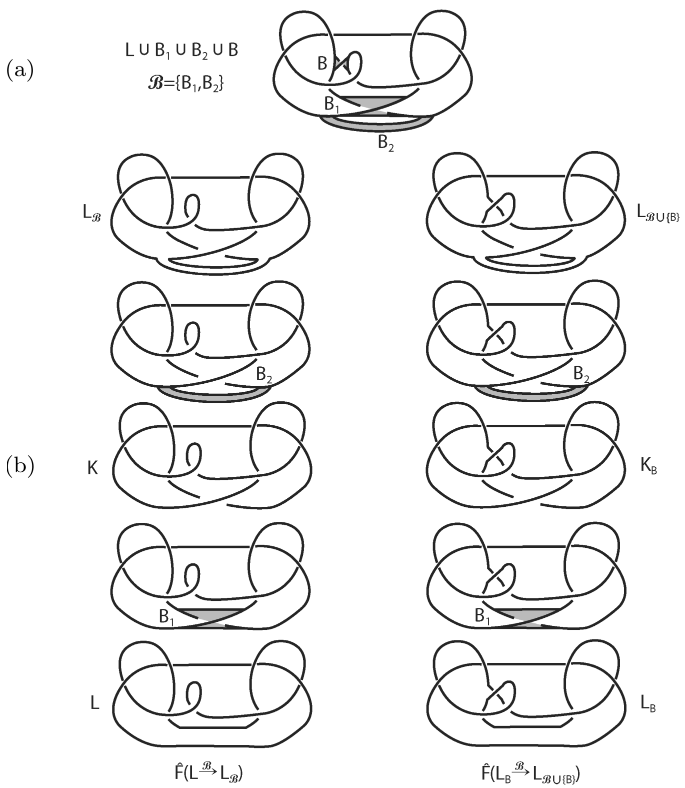


Fig. 16.

$$\begin{tikzcd} L \arrow[r, "\mathfrak{B}"] \arrow[d, "B"'] & L_{\mathfrak{B}} \arrow[d, "B"] \\ L_B \arrow[r, "\mathfrak{B}"] & L_{\mathfrak{B}\cup\{B\}} \end{tikzcd}$$

Fig. 17.

neither L_B nor $L_{\mathfrak{B}\cup\{B\}}$ are trivial since L_B and $L_{\mathfrak{B}\cup\{B\}}$ are Solomon's knots. In fact, the noncoherent band B is not standard. To obtain the closed realizing surface $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$, we need the condition that L_B and $L_{\mathfrak{B}\cup\{B\}}$ are trivial.

(3) The links L, $L_{\mathfrak{B}}$, L_B and $L_{\mathfrak{B}\cup\{B\}}$ have the diagrammatic relationship as depicted in Fig. 17.

(4) Let $\tau_n(K)$ denote Zeemann's n-twist-spin of a classical knot K. In [15], Satoh proved that $\sigma^n(K) \sharp \mathbf{R}P^2$ and $\sigma^{n+2}(K) \sharp \mathbf{R}P^2$ are ambient isotopic where $\sigma^n K$ denote the surface-knot obtained from $\tau_n(K)$ by surgery along a 1-handle. There is an open problem that $\tau_n(K) \sharp CC(\mathbf{R}P^2_\pm)$ and $\tau_{n+2}(K) \sharp CC(\mathbf{R}P^2_\pm)$ are ambient isotopic where $CC(\mathbf{R}P^2_\pm)$ is the $\pm$-cross-cap embedding of the standard real projective plane, see [1].

3. Triviality of Knotted 2-Knots

Theorem 2.3 say that the connected sum $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}}) \sharp \mathbf{R}P^2$ of a *knotted* 2-*knot* $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ and $\mathbf{R}P^2$ can be *unknotted* if the 2-knot $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ is trivial. The triviality of $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ is related with Fox–Hosokawa conjecture [5, 10, 12, 14].

Fox–Hosokawa conjecture. *If $\mathfrak{B}$ is a fusion band set attaching to the trivial link L and if $L_{\mathfrak{B}}$ is the unknot, then $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is trivial.*

It is known that Fox–Hosokawa conjecture is a special case of the following unknotting conjecture, see [12, 14].

Unknotting conjecture. *For a 2-knot F in $\mathbf{R}^4$, the complement $\mathbf{R}^4 - F$ is homotopy equivalent to S^1 if and only if $F \subset \mathbf{R}^4$ is unknotted.*

Notice that $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ in Fox–Hosokawa conjecture is a 2-knot. It is clear that if $\mathfrak{B}$ is a union of standard bands attaching to a trivial link L, then the 2-knot $\hat{F}(L \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}})$ is trivial. Lemma 1.7 says that Fox–Hosokawa conjecture is true when $\mathfrak{B}$ consists of exactly one fusion band.

If $\hat{F}(L_s \xrightarrow{\mathfrak{B}_s} K \xrightarrow{\mathfrak{B}_t} L_t)$ is a normal form of a 2-knot and if K is the unknot, then $\hat{F}(L_s \xrightarrow{\mathfrak{B}_s} K \xrightarrow{\mathfrak{B}_t} L_t)$ is the connected sum of $\hat{F}(L_s \xrightarrow{\mathfrak{B}_s} K)$ and $\hat{F}(K \xrightarrow{\mathfrak{B}_t} L_t)$. Hence the 2-knot $\hat{F}(L_s \xrightarrow{\mathfrak{B}_s} K \xrightarrow{\mathfrak{B}_t} L_t)$ is trivial provided that Fox–Hosokawa

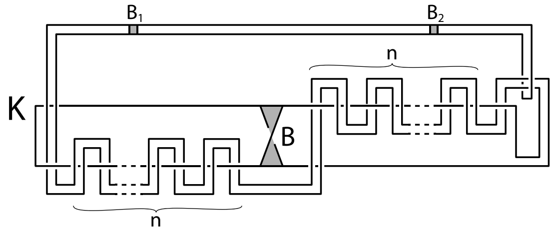


Fig. 18.

conjecture is true. In particular, if each of $\mathfrak{B}_\mathfrak{s}$ and $\mathfrak{B}_\mathfrak{t}$ consists of exactly one band, then $\hat{F}(L_s \xrightarrow{\mathfrak{B}_\mathfrak{s}} K \xrightarrow{\mathfrak{B}_\mathfrak{t}} L_t)$ is always trivial. Indeed, the triviality of the closed realizing surface $\hat{F}(L_B \xrightarrow{\mathfrak{B}} L_{\mathfrak{B}\cup\{B\}})$ in Viro's example and the following example can be shown by this observation.

Example 3.1. Consider a ribbon knot K and the band sets $\mathfrak{B}_\mathfrak{s} = \{B_1\}$ and $\mathfrak{B}_\mathfrak{t} = \{B_2\}$ attaching to K in Fig. 18. Note that if B_1 is a fusion band, then B_2 is a fission band, and that both K_{B_1} and K_{B_2} are the trivial links with 2-components. Let F denote a 2-knot whose normal form is $\hat{F}(K_{B_1} \xrightarrow{B_1} K \xrightarrow{B_2} K_{B_2})$. Indeed, F is a ribbon 2-knot. The fundamental group of the complement of F is given as $\langle x, y \,|\, xy^{2n+2}xy^{-1}x^{-1}y^{-2n-2} = 1\rangle$ so that the Alexander polynomial of F is $t^{2n+3} - t^{2n+2} + 1$. Hence F is nontrivial.

On the other hand, since K_B is the unknot, $\hat{F}(K_{\{B_1,B\}} \xrightarrow{B_1} K_B \xrightarrow{B_2} K_{\{B_2,B\}})$ is unknotted. By Theorem 2.3, $F \sharp \mathbf{R}P^2_+$ is ambient isotopic to $\mathbf{R}P^2_+$.

Finally, suppose that there is a complete fusion band set $\mathfrak{B}$ attaching to the trivial link. Notice that $\mathfrak{B}$ is not standard in general if $\mathfrak{B}$ consists of more than one band, as seen in Fig. 4. Scharlemann and Thompson conjectured that $\mathfrak{B}$ can be changed into a standard band set by a finite number of band slidings and band passes, depicted in Fig. 19. It is known [10] that Scharlemann–Thompson conjecture implies Fox–Hosokawa conjecture.

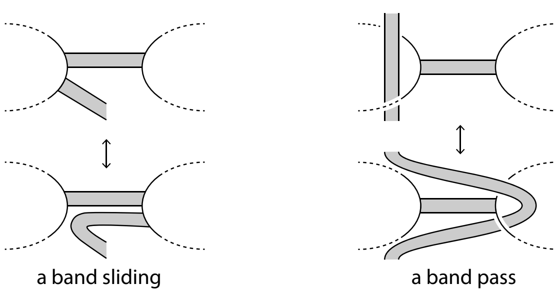


Fig. 19.

Acknowledgments

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References

[1] Y. Bae, S. Carter, S. Choi and S. Kim, Non-orientable surfaces in 4-dimensional space, *J. Knot Theory Ramifications* **23**(11) (2014) 52 pp.
[2] S. Bleiler and M. Scharlemann, A projective plane in $\mathbf{R}^4$ with three critical points is standard. Strongly invertible knots have property P, *Topology* **27**(4) (1988) 519–540.
[3] S. Carter, S. Kamada and M. Saito, *Surfaces in* 4*-sapce* (Springer, 2000).
[4] F. Hosokawa, A concept of cobordism between links, *Ann. of Math.* (2) **86**(2) (1967) 362–373.
[5] F. Hosokawa, On trivial 2-spheres in 4-space, *Quart. J. Math.* **19** (1968) 249–256.
[6] F. Hosokawa and A. Kawauchi, Proposals for unknotted surfaces in four-spaces, *Osaka J. Math.* **16**(1) (1979) 233–248.
[7] J. Howie and H. Short, The band sum problem, *J. London Math. Soc.* **31** (1985) 572–576.
[8] S. Kamada, Non-orientable surfaces in 4-space, *Osaka J. Math.* **26** (1989) 367–385.
[9] S. Kamada, *Braid and Knot Theory in Dimension Four* (American Mathematical Society, Providence, RI, 2002).
[10] A. Kawauchi, Imitations of $(3,1)$-dimensional manifold pairs, *Sügaku* **40** (1988) 193–204 (in Japanese); *Sugaku Expositions* **2** (1989) 141–156.
[11] A. Kawauchi, Almost identical imitations of $(3,1)$-dimensional manifold pairs, *Osaka J. Math.* **26** (1989) 743–758.
[12] A. Kawauchi, A Survey of Knot Theory (Birkhäuser, 1996).
[13] A. Kawauchi, T. Shibuya and S. Suzuki, Descriptions on surfaces in four-space I, *Math. Semin. Notes Kobe Univ.* **11** (1982) 75–125.
[14] A. Kawauchi, T. Shibuya and S. Suzuki, Descriptions on surfaces in four-space II, *Math. Semin. Notes Kobe Univ.* **11** (1983) 31–69.
[15] S. Satoh, Non-additivity for triple point numbers on the connected sum of surface-knots, *Proc. Amer. Math. Soc.* **133** (2004) 613–616.
[16] M. Scharlemann, Smooth spheres in $\mathbf{R}^4$ with four critical points are standard, *Invent. Math.* **79** (1985) 125–141.
[17] O. Ja. Viro, Local knotting of submanifolds, *Math. USSR-Sb.* **19**(2) (1973) 166–176.
[18] H. Whitney, On the topology of differentiable manifolds, in *Lectures in Topology* (University of Michigan Press, 1940).